Example 3

The force required to stretch a spring x units beyond its natural length of 14 cm is found to be $F(x) = 25x$. How much work is done in stretching the spring from its natural length to a length of 18 cm?

In previous examples, we found the limiting sum of thin rectangular areas of the form

$$\underbrace{f(x^*)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}}$$

(Recalling, of course, that Area = (height) × (width))

by finding a definite integral of the form $\int_a^b f(x)\,dx$

Note: there is more than one way to choose x^* in the Riemann Sum. 3 possible choices are worked out in purple, green, and blue below. Of course, they all lead to the same conclusion...

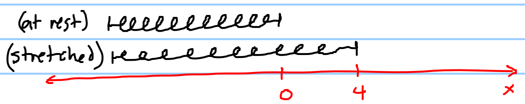
For a constant force, know that

Work = (Force) × (Displacemt)

So, in the same way, we wish to find the limiting sum of small amounts of work done that take the form:

$$\underbrace{F(x^*)}_{\text{force}} \cdot \underbrace{\Delta x}_{\text{displacement}}$$

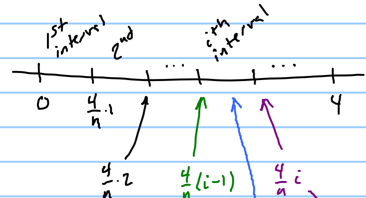
As such, we seek to find:



$$\int_0^4 F(x)\,dx = \int_0^4 25x\,dx$$

place spring on x-axis so end is at 0 when at rest.

Note $\Delta x = \frac{4-0}{n} = \frac{4}{n}$



midpoint (average of left/right endpoints)

$$\frac{\frac{4}{n}(i-1) + \frac{4}{n}i}{2} = \frac{4}{n}i - \frac{2}{n}$$

If x^* is chosen to be the left endpoint of each subinterval, then we have...

$$\int_0^4 F(x)\,dx = \int_0^4 25x\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 25\left(\frac{4}{n}(i-1)\right)\cdot\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{400}{n^2} \sum_{i=1}^{n} (i-1)$$

$$= \lim_{n\to\infty} \frac{400}{n^2}\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{400}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$

$$= \lim_{n\to\infty} \left[\frac{400\,n(n+1)}{2n^2} - \frac{400\,n}{n^2}\right]$$

$$= 200 - 0 = \boxed{200}$$

If x^* is chosen to be the midpoint of each subinterval, then we have...

$$\int_0^4 F(x)\,dx = \int_0^4 25x\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 25\left(\frac{4}{n}i - \frac{2}{n}\right)\cdot\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{200}{n^2} \sum_{i=1}^{n} (2i-1)$$

$$= \lim_{n\to\infty} \frac{200}{n^2}\left[2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{200}{n^2}\left[2\,\frac{n(n+1)}{2} - n\right]$$

$$= \lim_{n\to\infty} \left[\frac{200\,(n+1)}{n} - \frac{200}{n}\right]$$

$$= 200 - 0 = \boxed{200}$$

If x^* is chosen to be the right endpoint of the i^{th} subinterval then we have...

$$\int_0^4 F(x) = \int_0^4 25x\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 25\left(\frac{4}{n}i\right)\cdot\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{400}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{400}{n^2}\cdot\frac{n(n+1)}{2} = \boxed{200}$$

Other choices for x^* could be made – leading to the same result.

One should note, however, that for polynomial functions, the right endpoint is frequently easier algebraically.