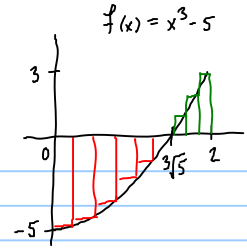
Example 2



Find the area bound by the curve $y = x^3 - 5$ and the x-axis between $x = 0$ to $x = 2$.

Note: the area of each rectangle above the x-axis takes the form:

$$f(x^*) \cdot \Delta x$$
(height) $\cdot$ (width)

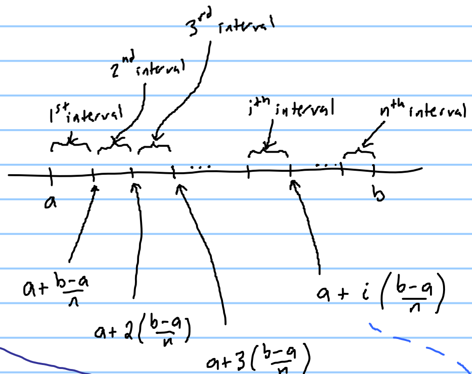
But note, $f(x_i^*)\Delta x_i$ will be negative when $x_i^* < \sqrt[3]{5}$.

So area sought should be:

$$\underbrace{-\int_0^{\sqrt[3]{5}} x^3 - 5\,dx}_{\text{area below}} + \underbrace{\int_{\sqrt[3]{5}}^{2} x^3 - 5\,dx}_{\text{area above}}$$

So we only have to do this once, let us find $\int_a^b x^3 - 5\,dx$ for any given a and b ...

Regular partition:



So $\Delta x = \frac{b-a}{n}$

Choosing x_i^* to be the right endpoint of the i^{th} interval, we have:

$$x_i^* = a + i\left(\frac{b-a}{n}\right)$$

$$\int_a^b x^3 - 5\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(a + i\left(\frac{b-a}{n}\right)\right)^3 - 5\right]\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a^3 + 3a^2\left(\frac{b-a}{n}\right)i + 3a\left(\frac{b-a}{n}\right)^2 i^2 + \left(\frac{b-a}{n}\right)^3 i^3 - 5\right]\left(\frac{b-a}{n}\right)$$

Recall: $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left[(a^3 - 5) + 3a^2\left(\frac{b-a}{n}\right)i + 3a\left(\frac{b-a}{n}\right)^2 i^2 + \left(\frac{b-a}{n}\right)^3 i^3\right]$$

Recall: $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[\sum_{i=1}^{n} (a^3 - 5) + \sum_{i=1}^{n} 3a^2\left(\frac{b-a}{n}\right)i + \sum_{i=1}^{n} 3a\left(\frac{b-a}{n}\right)^2 i^2 + \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)^3 i^3\right]$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[(a^3 - 5)\sum_{i=1}^{n} 1 + 3a^2\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} i + 3a\left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i^2 + \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^3\right]$$

Recall: $\sum_{i=1}^{n} 1 = n$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

and so on...

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[(a^3 - 5)n + 3a^2\left(\frac{b-a}{n}\right)\frac{n(n+1)}{2} + 3a\left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6} + \left(\frac{b-a}{n}\right)^3 \frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} (b-a)\left[(a^3 - 5) + 3a^2(b-a)\cdot\frac{n(n+1)}{2n^2} + 3a(b-a)^2\cdot\frac{n(n+1)(2n+1)}{6n^3} + (b-a)^3\cdot\frac{n^2(n+1)^2}{4n^4}\right]$$

$$= (b-a)\left[(a^3-5) + \frac{3a^2(b-a)}{2} + a(b-a)^2 + \frac{(b-a)^3}{4}\right]$$

So $\displaystyle\int_0^{\sqrt[3]{5}} x^3-5\,dx = \sqrt[3]{5}\left[-5 + 0 + 0 + \tfrac{5}{4}\right] = -5\sqrt[3]{5} + \tfrac{5}{4}\sqrt[3]{5}$

$$= -\frac{15}{4}\sqrt[3]{5}$$

and $\displaystyle\int_{\sqrt[3]{5}}^{2} x^3-5\,dx = (2-\sqrt[3]{5})\left[0 + \frac{3\sqrt[3]{25}(2-\sqrt[3]{5})}{2} + \sqrt[3]{5}(2-\sqrt[3]{5})^2 + \frac{(2-\sqrt[3]{5})^3}{4}\right]$

$$= (2-\sqrt[3]{5})\left[3\sqrt[3]{25} - \tfrac{3}{2}(5) + \sqrt[3]{5}(4-4\sqrt[3]{5}+\sqrt[3]{25}) + \frac{8-12\sqrt[3]{5}+6\sqrt[3]{25}-5}{4}\right]$$

$$= (2-\sqrt[3]{5})\left[3\sqrt[3]{25} - \tfrac{15}{2} + 4\sqrt[3]{5} - 4\sqrt[3]{25} + 5 + 2 - 3\sqrt[3]{5} + \tfrac{3}{2}\sqrt[3]{25} - \tfrac{5}{4}\right]$$

$$= (2-\sqrt[3]{5})\left[\tfrac{1}{2}\sqrt[3]{25} + \sqrt[3]{5} - \tfrac{7}{4}\right]$$

$$= \sqrt[3]{25} + 2\sqrt[3]{5} - \tfrac{7}{2} - \tfrac{5}{2} - \sqrt[3]{25} + \tfrac{7}{4}\sqrt[3]{5}$$

$$= \frac{15}{4}\sqrt[3]{5} - 6$$

Recall, the area we seek is: $\displaystyle -\int_0^{\sqrt[3]{5}} x^3-5\,dx + \int_{\sqrt[3]{5}}^{2} x^3-5\,dx$

$$= \frac{15}{4}\sqrt[3]{5} + \left(\frac{15}{4}\sqrt[3]{5} - 6\right)$$

$$= \boxed{\frac{15}{2}\sqrt[3]{5} - 6}$$