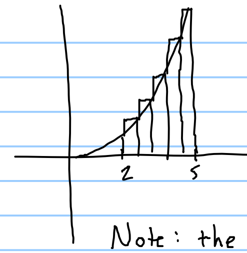
Example 1

> Find the area under the curve $y = x^2$ from $x = 2$ to $x = 5$.



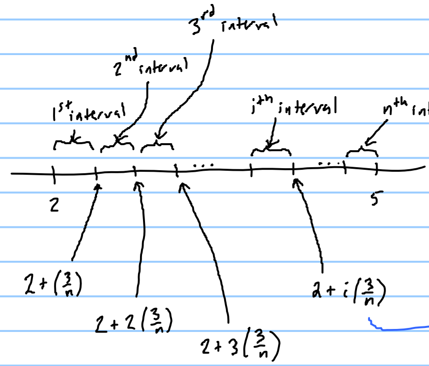
Note: the area of each rectangle takes the form:

$f(x^*) \cdot \Delta x$

(height) $\cdot$ (width)

$\int_2^5 x^2\,dx = ?$

Regular partition:



Note: $\underbrace{\Delta x = \frac{5-2}{n}}= \frac{3}{n}$

Let x_i^* be the right endpoint of the ith sub interval.

So $x_i^* = 2 + i\left(\frac{3}{n}\right)$

$$\int_2^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Recall $f(x) = x^2$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[2 + i\left(\frac{3}{n}\right)\right]^2 \cdot \left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 + 4\left(\frac{3}{n}\right)i + \left(\frac{9}{n^2}\right)i^2\right]\left(\frac{3}{n}\right)$$

Recall: $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[4 + \left(\frac{12}{n}\right)i + \left(\frac{9}{n^2}\right)i^2\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\sum_{i=1}^{n} 4 + \sum_{i=1}^{n}\left(\frac{12}{n}\right)i + \sum_{i=1}^{n}\left(\frac{9}{n^2}\right)i^2\right]$$

Recall: $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

$$= \lim_{n\to\infty} \frac{3}{n}\left[4 \cdot \sum_{i=1}^{n} 1 + \frac{12}{n}\sum_{i=1}^{n} i + \frac{9}{n^2}\sum_{i=1}^{n} i^2\right]$$

Recall:

$\sum_{i=1}^{n} 1 = n$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

and so on...

$$= \lim_{n\to\infty} \frac{3}{n}\left[4 \cdot n + \frac{12}{n} \cdot \frac{n(n+1)}{2} + \frac{9}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[12 + \frac{18(n+1)}{n} + \frac{9(n+1)(2n+1)}{2n^2}\right]$$

$$= \lim_{n\to\infty} 12 + \lim_{n\to\infty} \frac{18(n+1)}{n} + \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{2n^2}$$

$$= 12 + 18 + 9$$

$$= \boxed{39}$$