(1a) Let $a,b,c,d \in \mathbb{Q}$ and consider two arbitrary elements in $\mathbb{Q}(\sqrt{3})$: $(a+b\sqrt{3})$ and $(c+d\sqrt{3})$:

(1b) identical to (1a) except change every 3 in that answer to a 7.

Is addition closed?

yes! $(a+b\sqrt{3})+(c+d\sqrt{3}) = (a+c)+(b+d)\sqrt{3}$
and $(a+c), (b+d) \in \mathbb{Q}$ as $\mathbb{Q}$ is closed under addition

Is multiplication closed?

yes! Note
$$(a+b\sqrt{3})\cdot(c+d\sqrt{3}) = ac + ad\sqrt{3} + bc\sqrt{3} + 3bd$$
$$= (ac+3bd) + (ad+bc)\sqrt{3}$$
and $(ac+3bd), (ad+bc) \in \mathbb{Q}$
as $\mathbb{Q}$ is closed under addition and multiplication.

Are addition and multiplication both commutative? are they both associative?

yes! $\mathbb{Q}(\sqrt{3})$ inherits this from $\mathbb{R}$.

Do additive and multiplicative identities exist?

yes! Note $0 = 0+0\sqrt{3}$
$1 = 1+0\sqrt{3}$
and $0,1 \in \mathbb{Q}$
and these are the additive and multiplicative identies for $\mathbb{R}$.

Do additive inverses exist?

yes! $(a+b\sqrt{3})+(-a-b\sqrt{3}) = 0$
and $-a, -b \in \mathbb{Q}$ when $a,b \in \mathbb{Q}$

Do multiplicative inverses exist for non-zero values?

yes! Note, provided $a+b\sqrt{3} \neq 0$:
$$\frac{1}{a+b\sqrt{3}} = \frac{1}{(a+b\sqrt{3})}\cdot\frac{(a-b\sqrt{3})}{(a-b\sqrt{3})}$$
$$= \frac{a-b\sqrt{3}}{a^2-3b^2} = \left(\frac{a}{a^2-3b^2}\right)+\left(\frac{-b}{a^2-3b^2}\right)\sqrt{3}$$
which lies in $\mathbb{Q}$ by closure of addition / mult./div for non-zero rationals

Hence, $\mathbb{Q}(\sqrt{3})$ forms a field!

(2a)

$$
\begin{array}{r}
a+b\sqrt{2}+c\sqrt{5}+d\sqrt{10} \\
+\quad e+f\sqrt{2}+g\sqrt{5}+h\sqrt{10} \\
\hline
(a+e)+(b+f)\sqrt{2}+(c+g)\sqrt{5}+(d+h)\sqrt{10}
\end{array}
\checkmark
$$

$$
\begin{array}{r}
(a+b\sqrt{2}+c\sqrt{5}+d\sqrt{10}) \\
\times\ (e+f\sqrt{2}+g\sqrt{5}+h\sqrt{10}) \\
\hline
ah\sqrt{10}+2bh\sqrt{5}+5ch\sqrt{2}+10dh \\
+\ ag\sqrt{5}+bg\sqrt{10}+5cg+5dg\sqrt{2} \\
+af\sqrt{2}+2bf+cf\sqrt{10}+2df\sqrt{5} \\
+ae+be\sqrt{2}+ce\sqrt{5}+de\sqrt{10} \\
\hline
\end{array}
$$

$$
\begin{aligned}
= & (10dh+5cg+2bf+ae) \\
& +(5ch+5dg+af+be)\sqrt{2} \\
& +(2bh+ag+2df+ce)\sqrt{5} \\
& +(ah+bg+cf+de)\sqrt{10} \quad \checkmark
\end{aligned}
$$

... the rest follows by closure of $\mathbb{Q}$ under addition / mult.

(2b)

$$
\begin{array}{r}
(a+b\sqrt[4]{2}+c\sqrt[4]{4}+d\sqrt[4]{8}) \\
+(e+f\sqrt[4]{2}+g\sqrt[4]{4}+h\sqrt[4]{8}) \\
\hline
(a+e)+(b+f)\sqrt[4]{2}+(c+g)\sqrt[4]{4}+(d+h)\sqrt[4]{8}
\end{array}
\checkmark
$$

$$
\begin{array}{r}
(a+b\sqrt[4]{2}+c\sqrt[4]{4}+d\sqrt[4]{8}) \\
\times\ (e+f\sqrt[4]{2}+g\sqrt[4]{4}+h\sqrt[4]{8}) \\
\hline
ah\sqrt[4]{8}+2bh+2ch\sqrt[4]{2}+2dh\sqrt[4]{4} \\
ag\sqrt[4]{4}+bg\sqrt[4]{8}+2cg+2dg\sqrt[4]{2} \\
af\sqrt[4]{2}+bf\sqrt[4]{4}+cf\sqrt[4]{8}+2df \\
ae+be\sqrt[4]{2}+ce\sqrt[4]{4}+de\sqrt[4]{8} \\
\hline
\end{array}
$$

$$
\begin{aligned}
& (2bh+2cg+2df+ae) \\
& +(2ch+2dg+af+be)\sqrt[4]{2} \\
& +(2dh+ag+bf+ce)\sqrt[4]{4} \quad \checkmark \\
& +(ah+bg+cf+de)\sqrt[4]{8}
\end{aligned}
$$

... the rest follows by closure of $\mathbb{Q}$ under addition / mult.

(3a) $$\frac{x(y+2)}{y\sqrt{y+2}} = \frac{x\cdot(y+2)^1}{y\,(y+2)^{1/2}} = \frac{x\,(y+2)^{1/2}}{y} = \boxed{\frac{x\sqrt{y+2}}{y} \quad \text{where } y \neq -2}$$

(3b) $$\frac{x^2(4x^2-6x+9)}{\sqrt[3]{x}\cdot\sqrt[4]{2x-3}} = \frac{x^2(2x-3)^2}{x^{1/3}\,(2x-3)^{1/4}} = x^{5/3}(2x-3)^{7/4} = \boxed{x\,(2x-3)\,\sqrt[3]{x^2}\cdot\sqrt[4]{(2x-3)^3} \quad \text{where } x\neq 0, \tfrac{3}{2}}$$

(3c) $$\frac{2x^2-8x+8}{\sqrt{x}+\sqrt{2}} = \frac{2(x^2-4x+4)}{\sqrt{x}+\sqrt{2}} = \frac{2(x-2)^2}{(\sqrt{x}+\sqrt{2})}\cdot\frac{(\sqrt{x}-\sqrt{2})}{(\sqrt{x}-\sqrt{2})}$$

$$= \frac{2(x-2)^{\cancel{2}}(\sqrt{x}-\sqrt{2})}{\cancel{(x-2)}} = \boxed{2(x-2)(\sqrt{x}-\sqrt{2})}$$

Note, we <u>don't</u> pick up a restriction on x beyond the implicit $x\geq 0$ as $\sqrt{x}+\sqrt{2}>0$ always.

(3d) $$\frac{x^3-1}{1-\sqrt{x}} = \frac{(x^3-1)}{(1-\sqrt{x})}\cdot\frac{(1+\sqrt{x})}{(1+\sqrt{x})} = \frac{(x^3-1)(1+\sqrt{x})}{(1-x)}$$

$$= \frac{\overset{(-1)}{\cancel{(x-1)}}(x^2+x+1)(1+\sqrt{x})}{\cancel{(1-x)}} = \boxed{-(x^2+x+1)(1+\sqrt{x}) \quad \text{where } x\neq 1}$$

(3e) $$\frac{x^3+1}{1+\sqrt[3]{x}} = \frac{(x^3+1)}{(1+\sqrt[3]{x})}\cdot\frac{(1-\sqrt[3]{x}+\sqrt[3]{x^2})}{(1-\sqrt[3]{x}+\sqrt[3]{x^2})} = \frac{(x^3+1)(1-\sqrt[3]{x}+\sqrt[3]{x^2})}{1+x}$$

$$= \frac{\cancel{(x+1)}(x^2-x+1)\,(1-\sqrt[3]{x}+\sqrt[3]{x^2})}{\cancel{1+x}} = \boxed{(x^2-x+1)(1-\sqrt[3]{x}+\sqrt[3]{x^2}) \quad \text{where } x\neq -1}$$

4a) $f(x)=\sqrt{x} \to \dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$$= \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})} = \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

if $h\neq 0$, this simplifies to $\dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

notably, this is itself defined at $h=0$ and equals $\dfrac{1}{2\sqrt{x}}$ at this h value

4b) $f(x)=\sqrt{9-x} \to \dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}$

$$= \left(\frac{\sqrt{9-x-h}-\sqrt{9-x}}{h}\right)\cdot\frac{(\sqrt{9-x-h}+\sqrt{9-x})}{(\sqrt{9-x-h}+\sqrt{9-x})}$$

$$= \frac{(\cancel{9}-\cancel{x}-h)-(\cancel{9}-\cancel{x})}{h(\sqrt{9-x-h}+\sqrt{9-x})} = \frac{-h}{h(\sqrt{9-x-h}+\sqrt{9-x})}$$

if $h\neq 0$, this simplifies to $\dfrac{-1}{\sqrt{9-x-h}+\sqrt{9-x}}$

notably, the above is defined when $h=0$ and equals $\dfrac{-1}{2\sqrt{9-x}}$ at this h value

4c) $f(x)=\dfrac{1}{\sqrt{x}} \longrightarrow \dfrac{f(x+h)-f(x)}{h} = \dfrac{\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)}{h}\cdot\dfrac{\sqrt{x+h}\cdot\sqrt{x}}{\sqrt{x+h}\sqrt{x}}$

$$= \frac{(\sqrt{x}-\sqrt{x+h})}{h\sqrt{x+h}\sqrt{x}}\cdot\frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})} = \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-h}{h\sqrt{x+h}\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})}$$

if $h\neq 0$ this simplifies to

$$\frac{-1}{\sqrt{x+h}\sqrt{x}\,(\sqrt{x}+\sqrt{x+h})}$$

notably this last expression is defined when $h=0$ and at this h value equals:

$$\frac{-1}{\sqrt{x}\cdot\sqrt{x}\,(\sqrt{x}+\sqrt{x})} = \frac{-1}{2x\sqrt{x}}$$

technically, $\dfrac{-1}{2|x|\sqrt{x}}$ unless operating under the same domain as $f(x)=\dfrac{1}{\sqrt{x}}$ which requires $x>0$

4d) $f(x)=x^{3/2}$

$$\dfrac{f(x+h)-f(x)}{h} = \frac{((x+h)^{3/2}-x^{3/2})}{h}\cdot\frac{((x+h)^{3/2}+x^{3/2})}{((x+h)^{3/2}+x^{3/2})}$$

$$= \frac{(x+h)^3-x^3}{h((x+h)^{3/2}+x^{3/2})} = \frac{(\cancel{x^3}+3x^2h+3xh^2+h^3)-\cancel{x^3}}{h((x+h)^{3/2}+x^{3/2})}$$

$$= \frac{3x^2h+3xh^2+h^3}{h((x+h)^{3/2}+x^{3/2})} = \frac{h(3x^2+3xh+h^2)}{h((x+h)^{3/2}+x^{3/2})}$$

if $h\neq 0$, this simplifies to $\dfrac{3x^2+3xh+h^2}{((x+h)^{3/2}+x^{3/2})}$

notably, this last expression is defined when $h=0$ and equals

$$\frac{3x^2+0+0}{x^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}$$

at this h value

(4e) $f(x) = \sqrt[3]{x}$

recall $(a-b)(a^2+ab+b^2) = a^3-b^3$

$$\frac{f(x+h)-f(x)}{h} = \frac{\left(\sqrt[3]{x+h}-\sqrt[3]{x}\right)}{h} \cdot \frac{\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}{\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

$$= \frac{(x+h)-x}{h\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

$$= \frac{h}{h\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

if $h \neq 0$, this simplifies to:

$$\frac{1}{\left(\sqrt[3]{(x+h)^2}+\sqrt[3]{x(x+h)}+\sqrt[3]{x^2}\right)}$$

notably, this last expression is defined at $h=0$ and with this h value equals

$$\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x^2}+\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{x^2}}$$

(4f) $f(x) = x+\sqrt{x}$

$$\frac{f(x+h)-f(x)}{h} = \frac{(\cancel{x}+h)+\sqrt{x+h}-(\cancel{x}+\sqrt{x})}{h}$$

$$= \frac{h+\sqrt{x+h}-\sqrt{x}}{h} = \frac{h}{h}+\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \frac{h}{h}+\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{h}{h}+\frac{(\cancel{x}+h)-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{h}{h}+\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

if $h \neq 0$ this simplifies to

$$1+\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

notably, the expression above is defined at $h=0$ and at this value, equals

$$1+\frac{1}{2\sqrt{x}}$$